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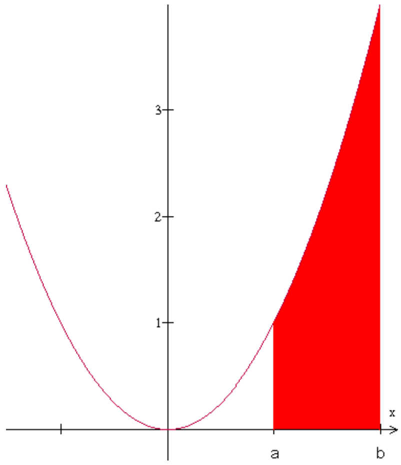
Calculus BC Project- Double Integral

Introduction

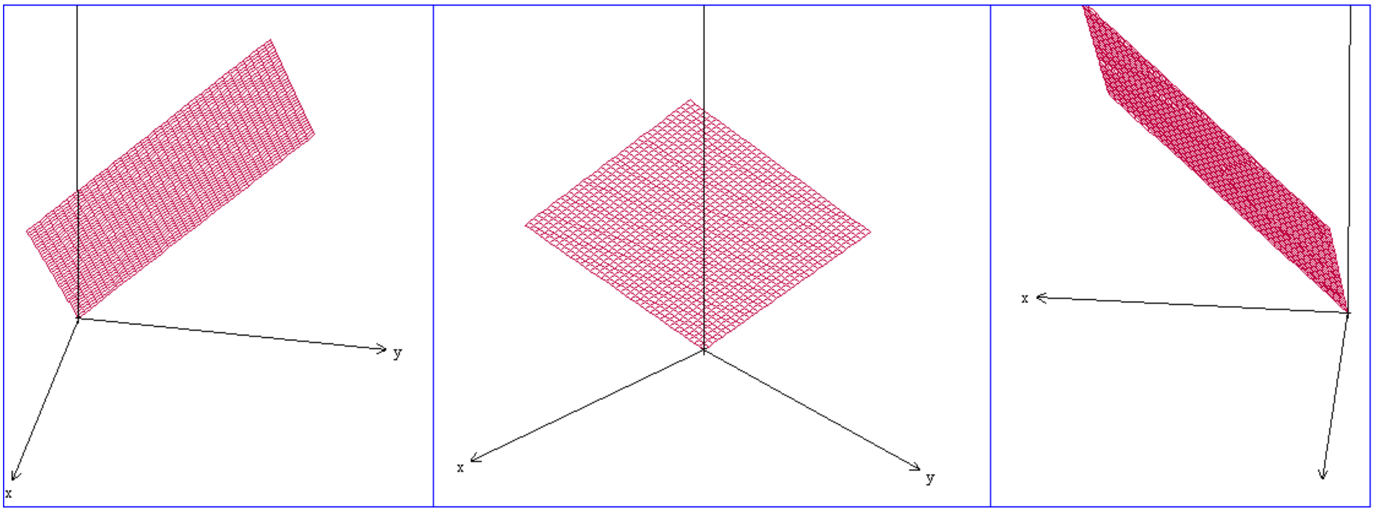


Integration is often used to find the area under a given curve. For example, the equation $\int_a^b x^2 dx$ gives the area under the curve of the equation $y = x^2$ from $[a,b]$.

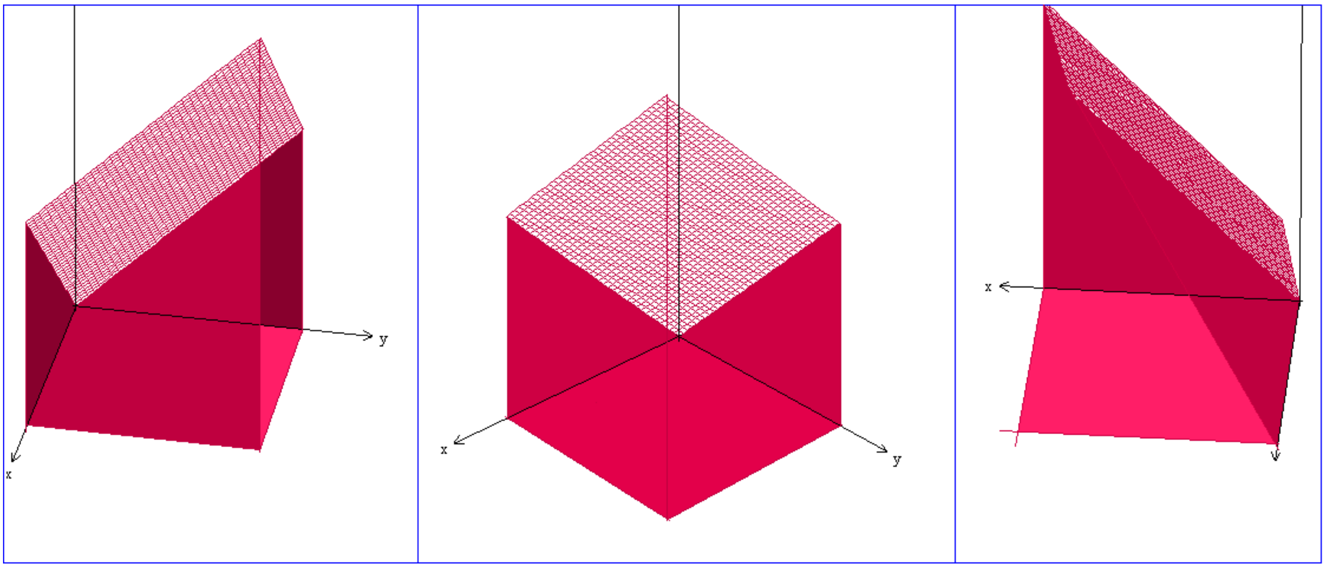
The equation given is given in terms of only one variable, x. Therefore the equation can be rewritten as $f(x)$. $f(x)$ gives a two dimensional view.

Double integral is used to find the volume. Double integral is used when the equation is in terms of two variables, x and y. The given equation is often written as $z = f(x,y)$. By doing so, a 3rd axis is portrayed and the graph is three dimensional. By using double integration, you do not find the area under the curve, bur rather the volume.

ex) $z = f(x,y) = x + y$
Below is the graph of $f(x,y)$, from three sides.



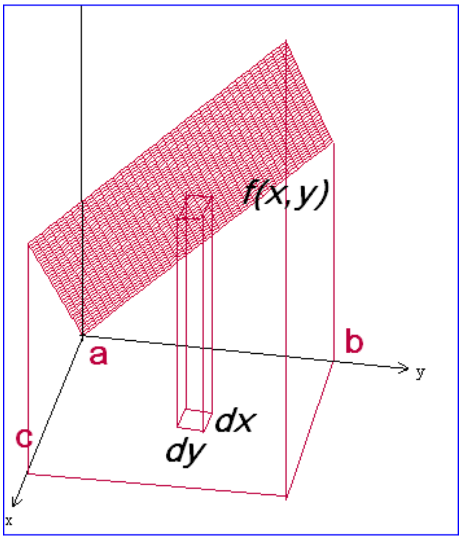
By using double integration, we will find the volume below the curve $f(x,y)$.



Proof 1:

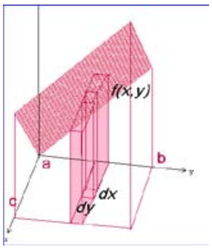
$f(x,y) = x + y$

Domain: $[a,c]$ range: $[a,b]$



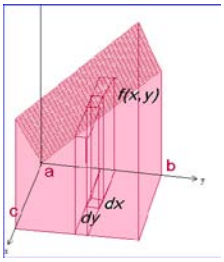
The single column is represented by

$f(x,y)dxdy$



$\int_a^c f(x,y)dxdy$ gives the volume of this figure.

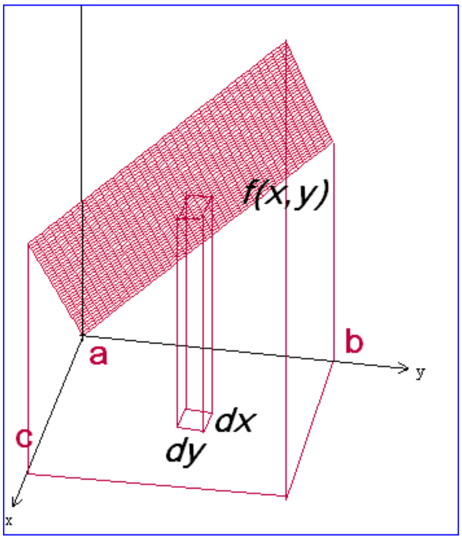
The columns are added up for one row from $[a,c]$.



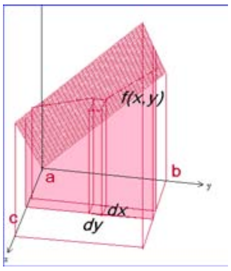
$\int_a^b \int_a^c f(x,y)dxdy$ gives the overall volume under the curve of $f(x,y)$.

All the volumes of the cross sections are added up.

Proof 2:

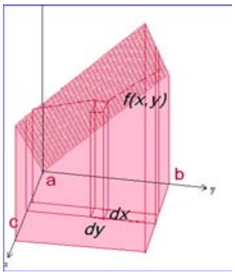


The single column is represented by
$f(x,y)dydx$



$\int_a^b f(x,y)dydx$ gives the volume of this figure.

The columns are added up for one row.



$\int_a^c\int_a^b f(x,y)dydx$ gives the overall volume under the curve of $f(x,y)$.

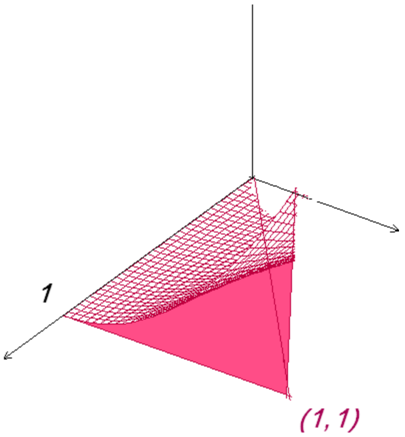
All the volumes of the cross sections are added up.

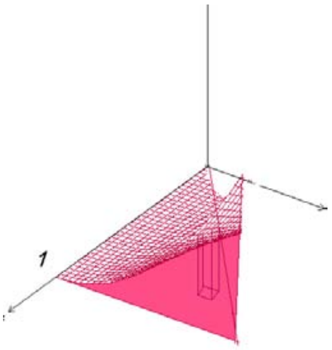
The volume of z can be found by both $\int_a^c\int_a^b f(x,y)dydx$ or $\int_a^b\int_a^c f(x,y)dxdy$ as the volume of one column can be found by both $f(x,y)dydx$ and $f(x,y)dxdy$.

Sometimes, $f(x, y)$ is bounded by variables instead of constants. Therefore, the area of the bottom section is not a rectangle.

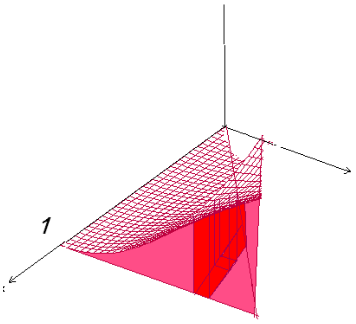
$z = xy^2$

$0 \le x \le 1 \qquad 0 \le y \le y$

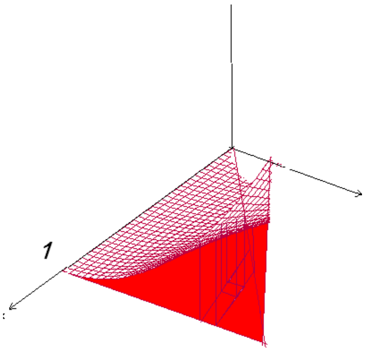




$xy^2 dxdy$ represents one column



$\int_1^y xy^2 dxdy$



$\int_0^1 \int_1^y xy^2 dxdy$

Solving Double integrals is simply like solving regular integrals. Remember what are you solving for (it may be in terms of *x* or *y*).

For ex)

$z = xy^2$

$x = [0,2] \qquad\qquad y = [0,1]$

Solution:

$$\int_0^1 \int_0^2 xy^2 dxdy$$

$$= \int_0^1 \frac{x^2}{2} y^2 \Big|_0^2 dy$$

$$= \int_0^1 2y^2 dy$$

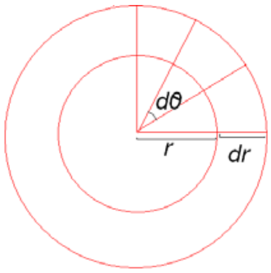
$$= 2\frac{y^3}{3}\Big|_0^1$$

$$= \frac{2}{3}$$

Solving Double Integral by Polar Coordinate

Double Integral is defined by $\iint_R f(r,\theta)dA = \iint_R f(r,\theta)rdrd\theta$. But why?

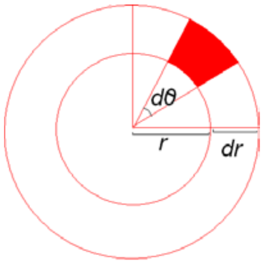
Instead of $f(x,y)$, polar coordinate is in terms of $f(r,\theta)$.



Area of a slice is determined by the formula $\frac{1}{2}d\theta r^2$.

This is given by $\pi r^2 \cdot \frac{\theta}{2\pi}$.

Area of a full circle is given by this formula because $\theta = 2\pi$



Therefore, the area of the shaded region (*dA)* is determined by

dA= outer slice- inner slice

$$dA = \frac{1}{2}d\theta\left(r+dr\right)^2 - \frac{1}{2}d\theta r^2$$
$$= \frac{1}{2}d\theta\left(r^2 + 2rdr + dr^2 - r^2\right)$$
$$= \frac{1}{2}d\theta\left(2rdr + dr^2\right)$$

dr is a very small number that approaches 0. Therefore, $dr^2 \approx 0$

$$dA = \frac{1}{2}d\theta\left(2rdr\right)$$
$$= rdrd\theta$$

Therefore,

Volume= $\iint_R f(r,\theta)dA = \int_{\theta_1}^{\theta_2}\int_{r=g_1(\theta)}^{r_2=g_2(\theta)} f(r,\theta)rdrd\theta$

Problem:

By using double integral, you can find volumes of 3D shapes.
Double integral will be used to find volumes of real life objects that are spherical and rectangular. At the same time, this will help support formulas that are used to find volumes.

Volumes are found by interpreting real life objects or things to a formula (*z*). By finding the formula z, double integration can be applied through a restricted area to find the volume under the curve.

This problem was selected as double integration gives brings forth the concept of 3D visualization. Through double integral, the idea of $f(x, y)$ and $f(r, \theta)$ is introduced and therefore a new z-axis. Double integral can be used to find and prove volume theories. This project will help teach and express how to solve double integrals through Cartesian and polar coordinate. Object's volumes are determined by double integral.

Application Problems

Solutions/Method/Results

i. **Show that by double integration, it is possible to find the volume of a sphere with radius 1.**

Volume of a sphere is often represented by $V = \frac{4}{3}\pi r^3$. This case, the volume of a sphere with radius 1 is $\frac{4}{3}\pi$ units^3. Double integration will provide an alternative way without using any given formulas.

Circle with radius 1 is represented by:

$x^2 + y^2 = 1$

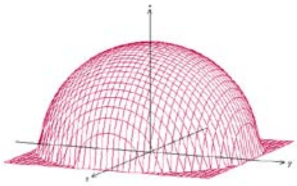
Therefore a sphere with radius 1 is represented by an additional variable (height) *z*.

$x^2 + y^2 + z^2 = 1$

$z = \pm\sqrt{1 - x^2 - y^2}$

In this case, we will find the volume of the sphere by only using

$z = +\sqrt{1 - x^2 - y^2}$ and multiplying by 2 as it would only represent half a sphere



Volume$= \iint_R \sqrt{1 - x^2 - y^2}\, dxdy$.   Transform to polar coordinate as it would be simpler to calculate.

Polar coordinate is simpler to calculate because you can remove $\cos^2\theta + \sin^2\theta = 1$.

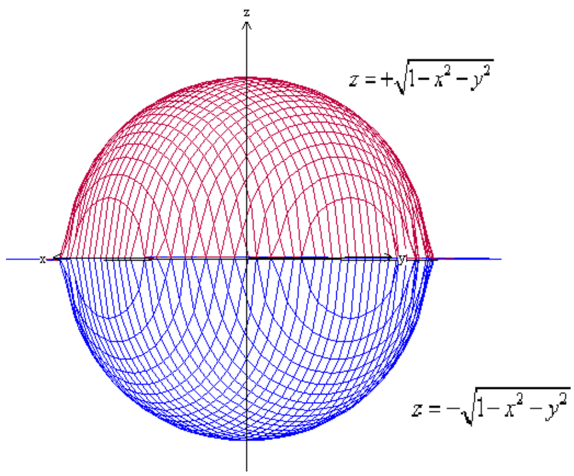
Recall:

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$R = \begin{cases} \theta & [0, 2\pi] \\ r & [0, 1] \end{cases}$$



Half of a sphere is defined by..

$$V = \iint_R \sqrt{1-\left(r\cos\theta\right)^2-\left(r\sin\theta\right)^2}\,dxdy$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2\cos^2\theta-r^2\sin^2\theta}\,rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2(\cos^2\theta+\sin^2\theta)}\,rdrd\theta \qquad ;\cos^2\theta+\sin^2\theta=1$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,rdrd\theta$$

$$let\ u=1-r^2 \qquad du=-2rdr$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_1^0 \sqrt{u}\,dud\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \sqrt{u}\,dud\theta$$

$$= \frac{1}{2}\frac{2}{3}\int_0^{2\pi} u^{\frac{3}{2}}\Big|_0^1\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{3}$$

Therefore, one complete sphere is $2\cdot\frac{2\pi}{3}=\frac{4\pi}{3}$

Test:

$$V=\frac{4\pi}{3}r^3; r=1$$

$$V=\frac{4\pi}{3}$$

The volume of the sphere with radius 1 is $\frac{4\pi}{3}$ units^3.

Application Problems

Solutions/Method/Results

ii. **<u>A large gumball machine has a radius of 72 inches. Find the maximum number of gumballs that can be place within the machine if each gumball has a diameter of 1 inch. Solve by double integral.</u>**

Volume of the large gum ball machine:

Circle with radius 72 is represented by:

$x^2 + y^2 = 72^2$

Therefore a sphere with radius 72 is represented by an additional variable (height) *z*.

$x^2 + y^2 + z^2 = 72^2$

$z = \pm\sqrt{72^2 - x^2 - y^2}$

In this case, we will find the volume of the sphere by only using $z = +\sqrt{72^2 - x^2 - y^2}$ and multiplying by 2 as it would only represent half a sphere

Volume= $2\iint_R \sqrt{72^2 - x^2 - y^2}\,dxdy$. Transform to polar coordinate as it would be simpler to calculate.

Polar coordinate is simpler to calculate because you can remove $\cos^2\theta + \sin^2\theta = 1$.

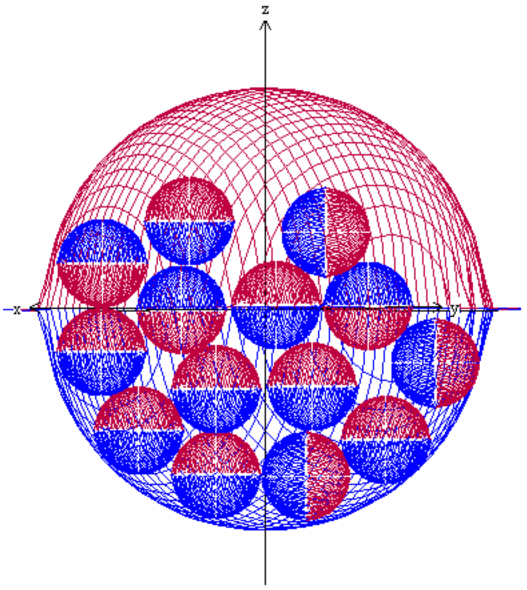
Recall:

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$R = \begin{cases} \theta & [0, 2\pi] \\ r & [0, 72] \end{cases}$$



$$V = 2\iint_R \sqrt{72^2 - (r\cos\theta)^2 - (r\sin\theta)^2}\,dxdy$$

$$= 2\int_0^{2\pi}\int_0^{72} \sqrt{72^2 - r^2\cos^2\theta - r^2\sin^2\theta}\,rdrd\theta$$

$$= 2\int_0^{2\pi}\int_0^{72} \sqrt{72^2 - r^2(\cos^2\theta + \sin^2\theta)}\,rdrd\theta \qquad ;\cos^2\theta + \sin^2\theta = 1$$

$$= 2\int_0^{2\pi}\int_0^{72} \sqrt{72^2 - r^2}\,rdrd\theta$$

$$let\ u = 72^2 - r^2 \qquad du = -2rdr$$

$$= 2\int_0^{2\pi} -\frac{1}{2}\frac{2}{3}\left(72^2 - r^2\right)^{\frac{3}{2}}\Big|_0^{72} d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi} -72^3 d\theta$$

$$= 248832\int_0^{2\pi} d\theta$$

$$= 248832\left(\theta\Big|_0^{2\pi}\right)$$

$$= 248832 \cdot 2\pi$$

$$= 497664\pi$$

Test:

$$V = \frac{4\pi}{3}r^3; r = 72$$

$$V = \frac{1492992\pi}{3} = 497664\pi$$

The volume of the sphere with radius 1 is $\frac{4\pi}{3}\text{units}^3$ from part i.

Amount of gumballs that the machine can withhold is approximately..

$$\frac{497664\pi}{\frac{4\pi}{3}} = 497664\pi\left(\frac{3}{4\pi}\right) \approx 373248$$

The machine can withhold about 373248 gumballs.

Application Problems

Solutions/Method/Results

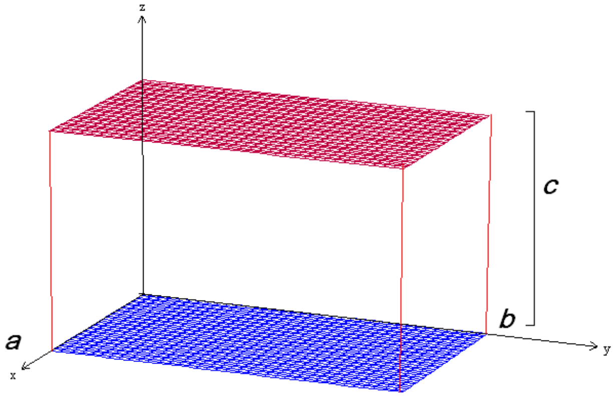
iii. **A rectangular prism has a width of a inches, length of *b* inches, and height of *c* inches. Prove that the volume is height x length x width (*abc*).**

Region *R* is defined by:

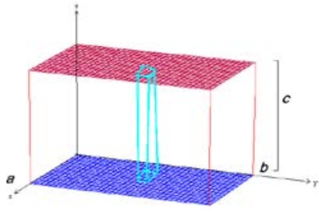
$x = [0, a]$

$y = [0, b]$

f(x,y)= c which is a constant



$$V = \int_0^b \int_0^a c\,dx\,dy$$



$$V = \int_0^b \int_0^a c\,dx\,dy$$

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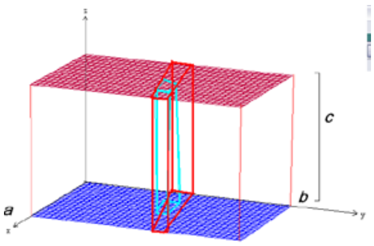
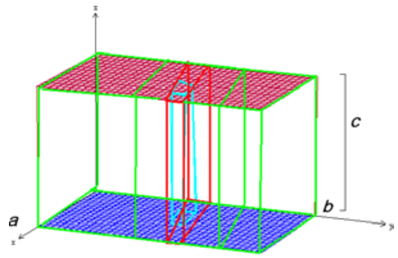




$$V = \int_0^b c \int_0^a dx\,dy$$

$$V = \int_0^b c\, x\Big|_0^a dy$$

$$V = c \int_0^b a\,dy$$

$$V = ca\, y\Big|_0^b$$

$$V = abc$$

By double integral, the volume of the box is found to be *abc.*

This shows that the volume formula for a box is correct as the formula is length x width x height.

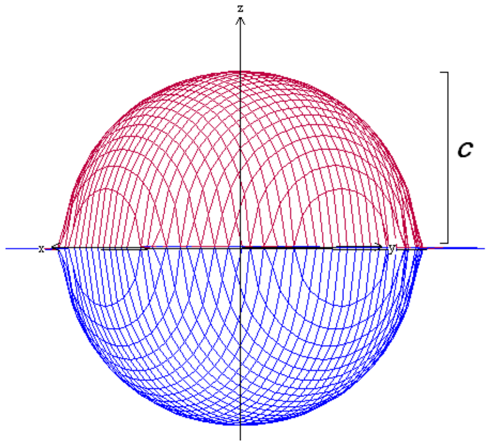
Application Problems

Solutions/Method/Results

iv. **Show that the volume of a sphere is $\frac{4}{3}\pi x^3$, where c represents the radius of the sphere by using double integral.**

a sphere with radius *r* is represented by an additional variable (height) *z*.

$$x^2 + y^2 + z^2 = c^2$$
$$z = \pm\sqrt{c^2 - x^2 - y^2}$$



Recall:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$

$$R = \begin{cases} \theta & [0, 2\pi] \\ r & [0, c] \end{cases}$$

$$V = 2\iint_R \sqrt{c^2 - (r\cos\theta)^2 - (r\sin\theta)^2}\,dxdy$$

$$= 2\int_0^{2\pi}\int_0^c \sqrt{c^2 - r^2\cos^2\theta - r^2\sin^2\theta}\,rdrd\theta$$

$$= 2\int_0^{2\pi}\int_0^c \sqrt{c^2 - r^2(\cos^2\theta + \sin^2\theta)}\,rdrd\theta \qquad ;\cos^2\theta + \sin^2\theta = 1$$

$$= 2\int_0^{2\pi}\int_0^c \sqrt{c^2 - r^2}\,rdrd\theta$$

$$let\ u = c^2 - r^2 \qquad du = -2rdr \qquad\qquad c\ is\ a\ number$$

$$= 2\int_0^{2\pi} -\frac{1}{2}\frac{2}{3}\left(c^2 - r^2\right)^{\frac{3}{2}}\Big|_0^c\, d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi} -c^3 d\theta$$

$$= \frac{2c^3}{3}\int_0^{2\pi} d\theta$$

$$= \frac{2c^3}{3}\left(\theta\Big|_0^{2\pi}\right)$$

$$= \frac{2c^3}{3}\cdot 2\pi$$

$$= \frac{4\pi c^3}{3} = \frac{4}{3}\pi c^3$$

Double integral supports that the volume formula for a sphere is true. The volume of a sphere is clearly found by using $\frac{4}{3}\pi c^3$, where *c* represents the radius.

Conclusion

The result of part i show that the volume of a sphere can be solved without using geometric formulas. Double integral provides an alternative way to find volume. At the same time, the volume is justified by using real volume formulas. This helps certify if the answer was correct or not. The volume of a sphere of radius 1 is $\frac{4}{3}\pi$ which is true as volume is represented by $\frac{4}{3}\pi r^3$.

The result of part ii shows that the maximum number of gumballs that can be put inside the gumball machine can be determined by using double integral. The total volume of the gumball machine that encloses the gumballs is determined by double integral. Each individual gumball's volume is found by using double integral from part 1 (as the gumball has a radius of 1). By calculating the total volume of the gumball machine and gumball itself, you can help determine how much can fit inside by dividing the volume of the gumball machine to the volume of one gumball. Decimals are ignored as a gumball should be represented by a whole number. Double integral is used to determine real life problems. It is used instead of volume formulas.

The result of part iii shows that the volume of a rectangular prism is found by multiplying base x length x height. Double integral is used to prove and support rectangular prism volume formulas. The result correctly shows and proves the theorem.

The results of part iv proves and supports the volume formula of a sphere. By double integral, it is found that the volume of a sphere is found by $\frac{4}{3}\pi c^3$; c =radius.

Overall, the project process was a success as new concepts were understood. Double integral introduces the idea of a 3rd axis which allows 3D graphs to be plot. Double integral also allows you to find and support volume formulas.

Areas of further study can be conducted to support and prove the volume of other 3D shapes. Other 3D shapes include cylinders, cones, and much more. While this project focuses on rectangular prisms and spheres only, much more other 3D shapes can be used.
Besides double integrals, further studies could take place on triple integrals. Triple integrals involve the function $f(x, y, z)$. Triple integrals could be used to find volumes of different shapes.