Andrew Chau
Kelly Li

Calculus BC

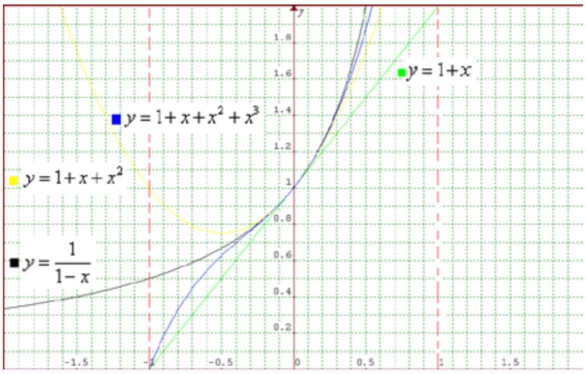
Date ______________

HW ______________

Introduction:

Brook Taylor was born on August 18, 1685 in Edmonton and died on November 30, 1731 in London. Taylor was an English mathematician best known for Taylor's theorem and the Taylor series.

Taylor polynomials are used to approximate functions that are difficult to evaluate. The Taylor series represents functions and the series is defined by an infinite sum of terms. As more terms are added, the function approaches closer to the function. For example, the function $y = \frac{1}{1-x}$ is defined by the Taylor series as $S = 1 + x + x^2 + \ldots$. The function and series approaches only when $|x| < 1$.



Taylor series allows a close approximation of the function when more terms are added. Taylor series is centered at any number besides zero; series centered at zero is called Maclaurin series.

Problem 1

a) Match the functions:

$$f_1 = 2xe^{-x} \qquad f_2 = \frac{1}{1-x} \qquad f_3 = \ln(1-x)$$

To the graphs in Figure 1 through 3 without using our calculator or computer and put the formulas under the corresponding graphs. Then check your conclusions by generating the curves on your calculator or computer with $-2 \le x \le 3, -3 \le y \le 3$.

$f_1 = 2xe^{-x}$

Domain: *all real numbers*

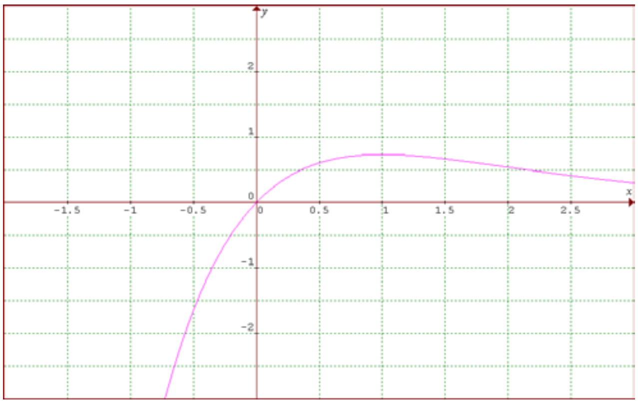
$f_1' = 2e^{-x} - 2xe^{-x}$

Let $x = 0$

$f_1'(0) = 2$

$f_1'(0) = 2$ means f is increasing at $x = 0$.

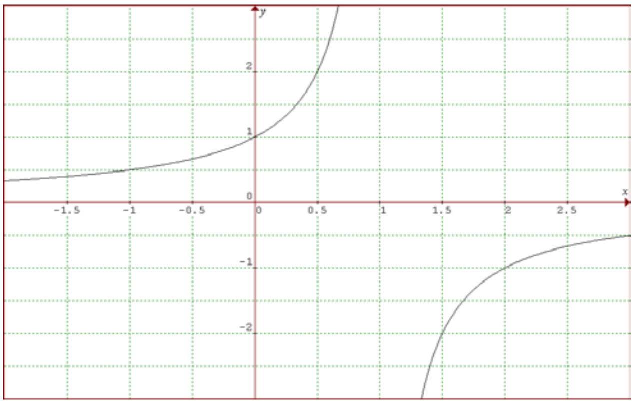
f_1 matches figure 1.



$f_2 = \dfrac{1}{1-x}$

Domain: *all real numbers except* $x = 0$

f_2 matches figure 3.



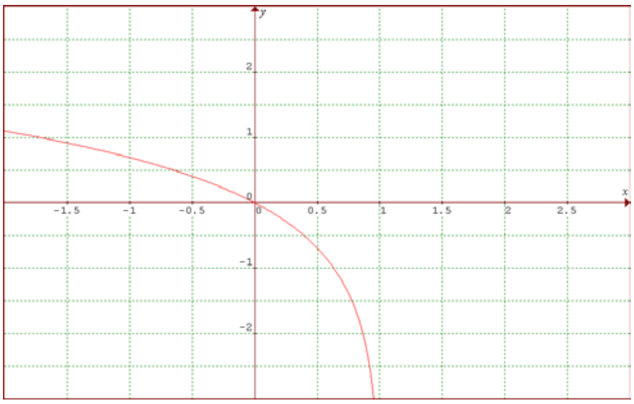
$f_3 = \ln(1-x)$

Domain: $1 - x > 0$

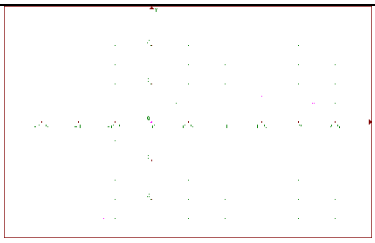
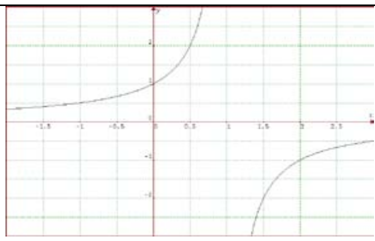
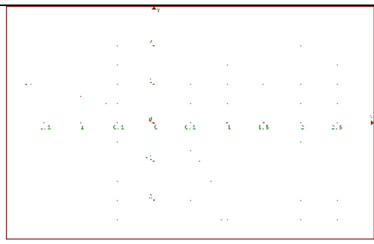
$-x > -1$

$x < 1$

f_3 matches figure 2.



b) *Generate the graphs of each of the following polynomials with the ranges of x and y from part (a) and determine which of the curves in Figures 1 through 3 each one best approximates near $x = 0$. Regenerate it with the graph it approximates, copy it in the corresponding figure, and put its formula under the graphs.*

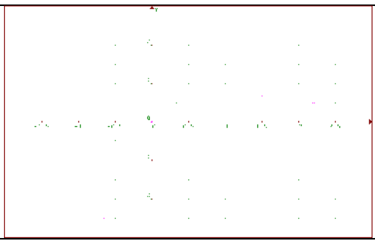
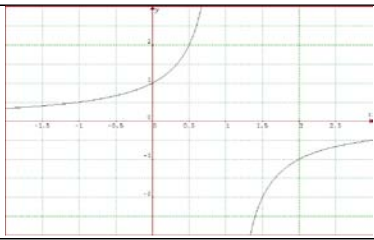
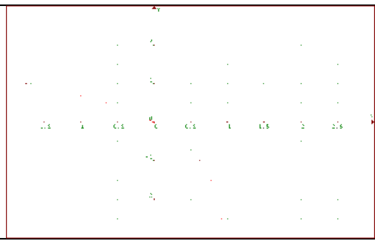
		
$f = f_1 = 2xe^{-x}$ $f_1(0) = 0$ $[f_1(0)]^+ > 0, [f_1(0)]^- < 0$	$f = f_3 = \ln(1-x)$ $f_3(0) = 0$ $[f_3(0)]^+ < 0, [f_3(0)]^- > 0$	$f = f_2 = \dfrac{1}{1-x}$ $f_2(0) = 1$

c) $Q_1 = 1 + x + x^2 + x^3 \qquad Q_2 = 2x - 2x^2 + x^3 \qquad Q_3 = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$

Show that each of the polynomials Q_1, Q_2, and Q_3 has the same value and the same first three derivatives at $x = 0$ as the corresponding function from part (a).

$Q_1 = 1 + x + x^2 + x^3$ $Q_1' = 1 + 2x + 3x^2 ; Q_1'(0) = 1$ $Q_1'' = 2 + 6x ; Q_1''(0) = 2$ $Q_1''' = 6 ; Q_1'''(0) = 6$	$f_2 = \dfrac{1}{1-x}$ $f_2' = \dfrac{1}{(1-x)^2} ; f_2'(0) = 1$ $f_2'' = \dfrac{-2(1-x)(-1)}{(1-x)^4} = \dfrac{2-2x}{(1-x)^4} ; f_2''(0) = 2$ $f_2''' = \dfrac{-2(1-x)^4 - (2-2x)\left(4(1-x)^3(-1)\right)}{(1-x)^8} ; f_2'''(0) = 6$
The first 3 derivatives of Q_1 and f_2 are 1, 2, and 6.	

$Q_2 = 2x - 2x^2 + x^3$ $Q_2' = 2 - 4x + 3x^2; Q_2'(0) = 2$ $Q_2'' = -4 + 6x; Q_2''(0) = -4$ $Q_2''' = 6; Q_2''(0) = 6$	$f_1 = 2xe^{-x}$ $f_1' = 2e^{-x} + 2x\left(-e^{-x}\right) = 2e^{-x}(1-x);\ f_1'(0) = 2$ $f_1'' = -2e^{-x}(1-x) + 2e^{-x}(-1)$ $f_1'' = -2e^{-x} + 2xe^{-x} - 2e^{-x} = 2e^{-x}(-2+x);\ f_1''(0) = 4$ $f_1''' = -2e^{-x}(-2+x) + 2e^{-x};\ f_1'''(0) = 6$
The first 3 derivatives of Q_2 and f_1 are 2, -4, and 6.	
$Q_3 = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$ $Q_3' = -1 - x - x^2; Q_3'(0) = -1$ $Q_3'' = 1 - 2x; Q_3'' = 1$ $Q_3''' = -2; Q_3''' = -2$	$f_3 = \ln(1-x)$ $f_3' = \frac{-1}{1-x};\ f_3'(0) = -1$ $f_3'' = \frac{-(-1)(-1)}{(1-x)^2} = \frac{-1}{(1-x)^2} = -(1-x)^{-2};\ f_3''(0) = -1$ $f_3'' = 2(1-x)^{-3}(-1) = -2(1-x)^{-3};\ f_3'''(0) = -2$
The first 3 derivatives of Q_3 and f_3 are -1, -1, and -2.	

		
$f = f_1 = 2xe^{-x}$	$f = f_3 = \ln(1-x)$	$f = f_2 = \frac{1}{1-x}$
$Q = Q_2 = 2x - 2x^2 + x^3$	$Q = Q_3 = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$	$Q = Q_1 = 1 + x + x^2 + x^3$

Problem 2

a) *Match the functions:*

$$f_4 = \cos x \qquad f_5 = x\cos x \qquad f_6 = \sin x$$

To the graphs in Figure 4 through 6 without using our calculator or computer and put the formulas under the corresponding graphs. Then check your conclusions by generating the curves on your calculator or computer with $-6 \le x \le 6, -4 \le y \le 4$.

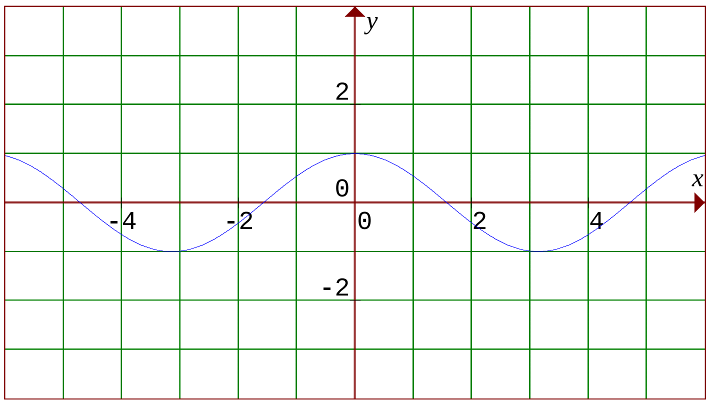
$f_4 = \cos x$

Domain: *all real numbers*

$f_4'(0) = -\sin 0 = 0$

Local extreme at $x = 0$.

f_4 matches figure 5.



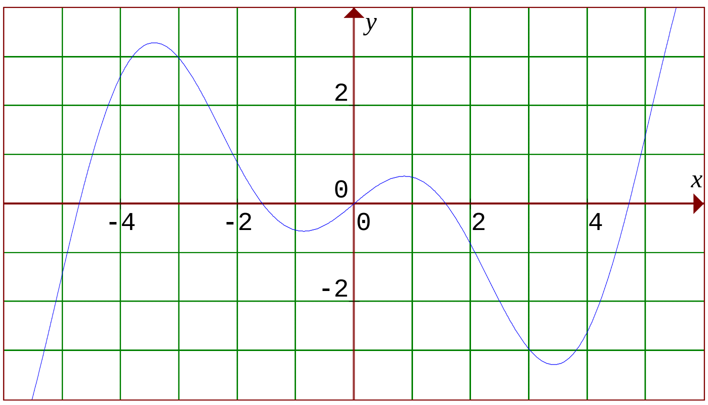
$f_5 = x\cos x$

Domain: *all real numbers*

$f_5'(0) = 1(\cos 0) + 0(-\sin 0) = 1$

f is increasing at $x = 0$.

f_5 matches figure 6.



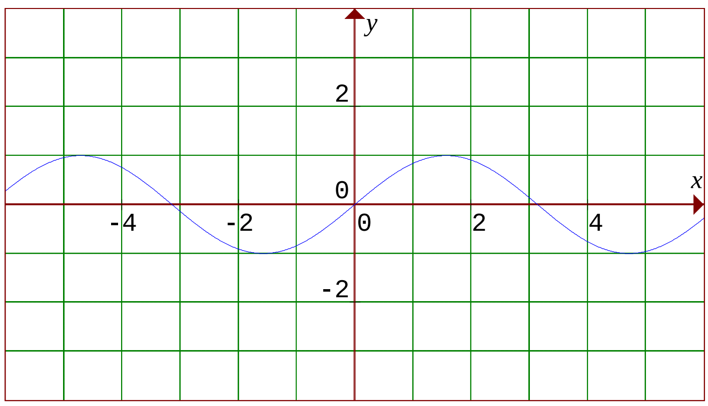
$f_6 = \sin x$

Domain: *all real numbers*

$f_6'(0) = \cos 0 = 1$

f is increasing at $x = 0$.

f_6 matches figure 4.

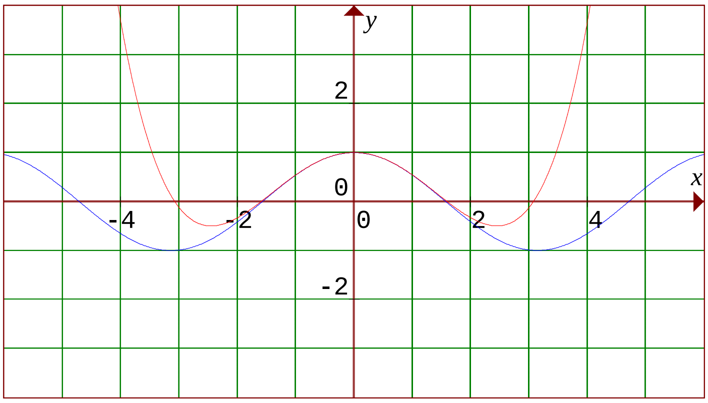


b) Generate the graphs of each of the following polynomials with the ranges of x and y from part (a) and determine which of the curves in Figures 4 through 6 each one best approximates near $x=0$. Regenerate it with the graph it approximates, copy it in the corresponding figure, and put its formula under the graphs.

$f_4 = \cos x$

$Q_6 = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$

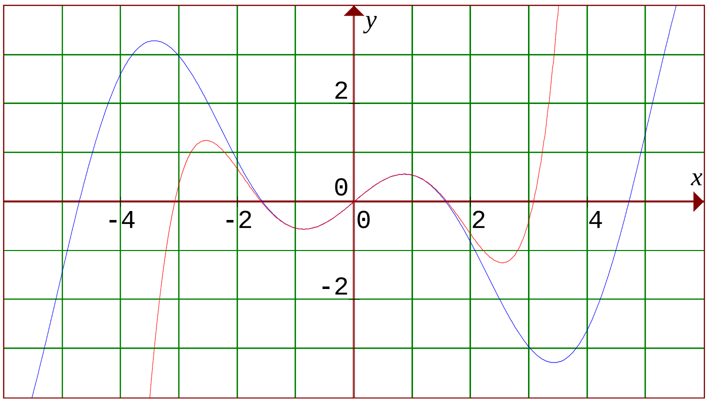
f_4 (blue) and Q_6 (red) overlap each other near the interval $-1 \le x \le 1$.



$f_5 = x\cos x$

$Q_4 = x - \frac{1}{2}x^3 + \frac{1}{4!}x^5$

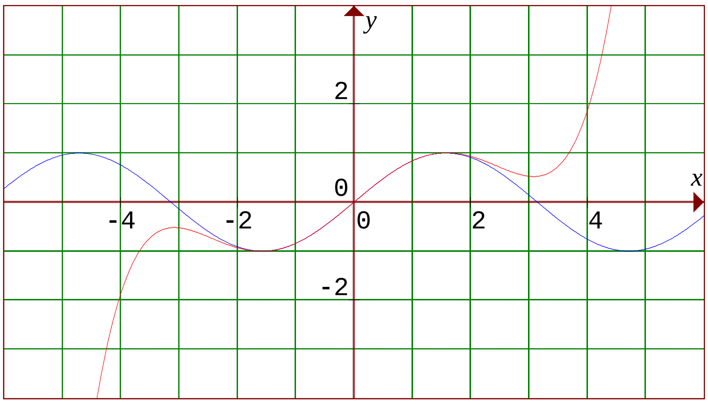
f_5 (blue) and Q_4 (red) overlap each other near the interval $-1 \le x \le 1$.



$f_6 = \sin x$

$Q_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

f_6 (blue) and Q_5 (red) overlap each other near the interval $-1 \le x \le 1$.



c) *Show that each of the polynomials* Q_4, Q_5*, and* Q_6 *has the same value and the same first three derivatives at* $x = 0$ *as the corresponding function from part (a).*

$Q_4 = x - \frac{1}{2}x^3 + \frac{1}{4!}x^5$ $Q_4' = 1 - \frac{3}{2}x^2 + \frac{5}{24}x^4;\ Q_4'(0) = 1$ $Q_4'' = -3x + \frac{5}{6}x^3;\ Q_4''(0) = 0$ $Q_4''' = -3 + \frac{15}{6}x^2;\ Q_4'''(0) = -3$	$f_5 = x\cos x$ $f_5' = \cos x + x(-\sin x);\quad f_5'(0) = 1$ $f_5'' = -\sin x - \sin x + x(-\cos x);\quad f_5''(0) = 0$ $f_5''' = -\cos x - \cos x - \cos x + (x)(\sin x);\quad f_5'''(0) = -3$
The first 3 derivatives of Q_4 and f_5 are 1, 0, and -3.	

$Q_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$ $Q_5' = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4;\ Q_5'(0) = 1$ $Q_5'' = -x + \frac{1}{6}x^3;\ Q_5''(0) = 0$ $Q_5''' = -1 + \frac{1}{2}x^2;\ Q_5'''(0) = -1$	$f_6 = \sin x$ $f_6' = \cos x;\quad f_6'(0) = 1$ $f_6'' = -\sin x;\quad f_6''(0) = 0$ $f_6''' = -\cos x;\quad f_6'''(0) = -1$
The first 3 derivatives of Q_5 and f_6 are 1, 0, and -1.	

$Q_6 = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$ $Q_6' = -x + \frac{1}{6}x^3;\ Q_6'(0) = 0$ $Q_6'' = -1 + \frac{1}{2}x^2;\ Q_6''(0) = -1$ $Q_6''' = x;\ Q_6'''(0) = 0$	$f_4 = \cos x$ $f_4' = -\sin x;\quad f_4'(0) = 0$ $f_4'' = -\cos x;\quad f_4''(0) = -1$ $f_4''' = \sin x;\quad f_4'''(0) = 0$
The first 3 derivatives of Q_6 and f_4 are 0, -1, and 0.	

Problem 3

Polynomials Q_1 *through* Q_6 *are called Taylor polynomial approximations of the corresponding functions* f_1 *through* f_6*. What general principle about Taylor polynomials do the results of Problems 1c and 2c suggest?*

The results of 1c and 2c suggests that both the function f and polynomial Q have the same derivatives (regardless of how many times you differentiate) if the interval of x is near 0.

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Calculus BC

Date _30/3/09_________
HW _Taylor Polynomials II_

Problem 1:

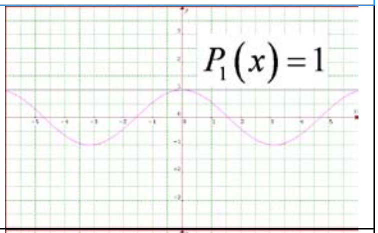
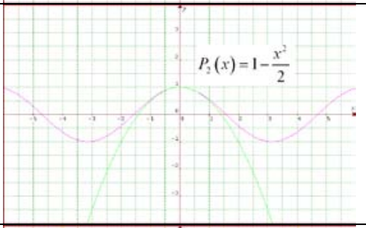
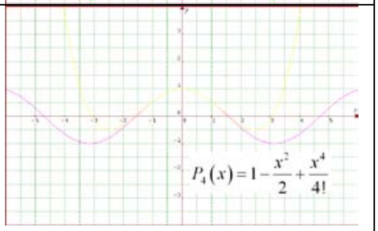
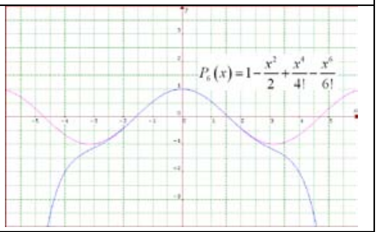
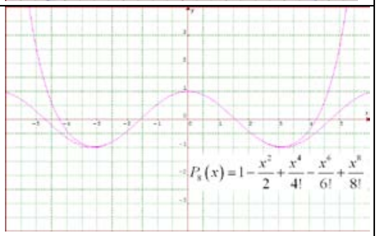
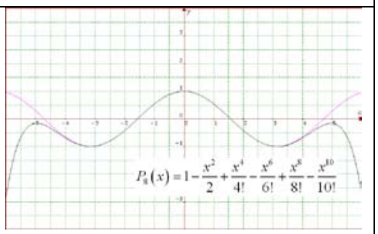
Figure 1 shows the graph of $\cos x$ and its Taylor polynomial $P_0(x)=1$. The first row of Table shows the values of $\cos(1)$, $P_0(1)$, and $\cos(1)-P_0(1)$. Find $P_2(x)$; generate its graph with the graph of $\cos x$ for $-6\le x\le 6, -4\le y\le 4$; copy it in Figure 2; and put the values of $P_2(1)$ and $\cos(1)-P_2(1)$ on the second row of the table, with three digit accuracy in column 3. Then repeat the process with P_4, P_4, P_4, and P_{10}, using Figures 3 through 6 and the other rows of the table. How does increasing n affect the approximations?

$$P_n(x)= f(0)+xf'(0)+\frac{x^2 f''(0)}{2!}+\ldots+\frac{x^n f^{(n)}(0)}{n!}$$

$$P_n(x)=1+0x-\frac{x^2}{2!}+\frac{0x^3}{3!}+\frac{x^4}{4!}+\frac{0x^5}{5!}-\frac{x^6}{6!}+\frac{0x^7}{7!}+\frac{x^8}{8!}+\frac{0x^9}{9!}-\frac{x^1}{10!}$$

$$P_n(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}$$

$y=\cos x$	$y(0)=1$		$y^{(6)}=-\cos x$	$y^{(6)}(0)=-1$
$y'=-\sin x$	$y'(0)=0$		$y^{(7)}=\sin x$	$y^{(7)}(0)=0$
$y''=-\cos x$	$y''(0)=-1$		$y^{(8)}=\cos x$	$y^{(8)}(0)=1$
$y'''=\sin x$	$y'''(0)=0$		$y^{(9)}=-\sin x$	$y^{(9)}(0)=0$
$y^{(4)}=\cos x$	$y^{(4)}(0)=1$		$y^{(10)}=-\cos x$	$y^{(10)}(0)=-1$
$y^{(5)}=-\sin x$	$y^{(5)}(0)=0$			

n	$\cos(1)$	$P_n(1)$	$\cos(1) - P_n(1)$	*Graph*
1	0.5403023050	$P_1(x) = 1$ $P_1(1) = 1$	-0.460	
2	0.5403023050	$P_2(x) = 1 - \frac{x^2}{2}$ $P_2(1) = 1 - \frac{1}{2} = \frac{1}{2}$	0.043	
4	0.5403023050	$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ $P_4(1) = 1 - \frac{1}{2} + \frac{1}{4!} = \frac{13}{24}$ $P_4(1) = 0.5416$	-1.36×10^{-3}	
6	0.5403023050	$P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ $P_6(1) = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} = \frac{389}{720}$ $P_6(1) = 0.5402$	2.45×10^{-5}	
8	0.5403023050	$P_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$ $P_8(1) = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = \frac{4357}{8064}$ $P_8(1) = 0.5403$	-2.74×10^{-7}	
10	0.5403023050	$P_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$ $P_8(1) = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!}$ $P_8(1) = 0.5403$	2.08×10^{-9}	

Increasing n causes $P_n(x)$ to approach the function $f(x)$. As n increases from one to ten in the above exercise, the differences between the actual function and $P_n(x)$ becomes less and less. Increasing n causes the Taylor polynomial to represent the actual function.

Problem 2:

Find the Taylor polynomial approximations $P_3(x)$ of $5+e^x$, centred at $x=0$. Generate the graphs of $5+e^x$ and $P_3(x)$ with $-5 \le x \le 5, -10 \le y \le 25,$ and y-scale = 5 and copy them in Figure 7. Then complete Table 2 of values of the two functions, with three digit accuracy in the last column. What do the graph and table suggest about the approximation for values of x that are close or far from 0?

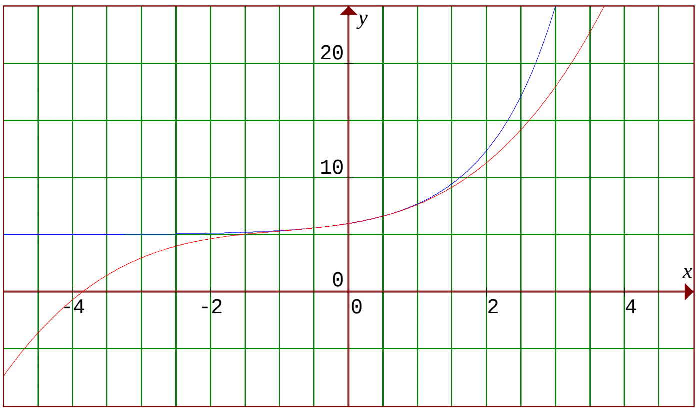
$$y = e^x + 5$$
$$y' = e^x$$
$$y'' = e^x$$
$$y'''' = e^x$$

$$P_3(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3$$
$$P_3(x) = e^0 + 5 + e^0 x + \frac{1}{2}e^0 x^2 + \frac{1}{6}e^0 x^3$$
$$P_3(x) = 6 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$y = 5 + e^x$

$y = P_3(x)$

$[-5, 5]$ x $[-10, 25]$:



x	$e^x + 5$	$P_3(x)$	$e^x + 5 - P_3(x)$
1	7.718	7.666	0.0516
0.1	6.105	6.105	0.00000425
0.01	6.010	6.010	0.000000000417
2	12.389	11.333	1.06
3	25.085	18	7.085
15	3269022.372	696	3268326.372

The graph and table suggests that the closer x is to zero, the more accurately $y = e^x + 5$ can be approximated through $P_3(x)$.

Find a Taylor Polynomial about zero of:

a. $y = \sin x$

$$P_n(x) = \sin 0 + x\cos 0 - \frac{x^2 \sin 0}{2!} - \frac{x^3 \cos 0}{3!} + \frac{x^4 \sin 0}{4!} + \frac{x^5 \cos 0}{5!} + \ldots + \frac{x^n f^n(0)}{n!}$$

$$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \bullet \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(2n+2)(2n+1)2n!} \bullet \frac{(2n+1)2n!}{(-1)^n x^{2n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{-x^2}{(2n+3)(2n+2)}\right| = 0 \prec 1$$

The series is convergent for $-\infty \prec x \prec \infty$.

b. $y = \cos x$

$$P_n(x) = 1 - x\sin 0 - \frac{x^2 \cos 0}{2!} + \frac{x^3 \sin 0}{3!} + \frac{x^4 \cos 0}{4!} - \frac{x^5 \sin 0}{5!} - \frac{x^6 \cos 0}{6!}$$
$$+ \frac{x^7 \sin 0}{7!} + \frac{x^8 \cos 0}{8!} - \frac{x^9 \sin 0}{9!} - \frac{x^{10} \cos 0}{10!} + \ldots + \frac{x^n f^{(n)}(0)}{n!}$$

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2n+2)!} \frac{(2n)!}{(-1)^n x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)(2n+1)2n!} \frac{2n!}{(-1)^n x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{-x^2}{(2n+2)(2n+1)} \right| = 0 \prec 1$$

The series is convergent for $-\infty \prec x \prec \infty$.

c. $y = e^x$

$$P_n(x) = e^0 + e^0 x + \frac{e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + \ldots + \frac{e^0 x^n}{n!}$$

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{(n+1)n!} \frac{n!}{1} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{(n+1)} \right| = 0 \prec 1$$

The series is convergent for $-\infty \prec x \prec \infty$.

d. $y = \tan^{-1} x$

$$y' = \frac{1}{1+x^2}$$

Geometric series with $r = -x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n}$$

$$\int \frac{dt}{1+x^2} = \int 1 - t^2 + t^4 - t^6 + \ldots + (-1)^n t^{2n} dt$$

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots + (-1)^n \frac{t^{2n+1}}{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1} \frac{2n+1}{(-1)^n x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{-x^2 2n+1}{2n+3} \right|$$

$$\lim_{n\to\infty} \left| \frac{-x^2 (2n+1)}{(2n+3)} \right| = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty} \left| \frac{-2x^2}{2} \right| = -x^2$$

$$\left| -x^2 \right| \prec 1$$

$$\left| x^2 \right| \prec 1$$

$$0 \prec x^2 \prec 1$$

$$0 \prec x \prec 1$$

If $x = 0$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0$$

Alternating series (Convergent)

If $x = 1$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$\lim_{x\to\infty}\frac{1}{2n+1}=0$$

Alternating series (Convergent)

The series $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ is convergent for $0\le x\le 1$.

e. $y=e^{x^2}$

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}=1+\frac{x^2}{1!}+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots+\frac{x^{2n}}{n!}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(n+1)!}\frac{n!}{x^{2n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)n!}\frac{n!}{x^{2n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(n+1)}\right|=0\prec 1$$

The series is convergent for $-\infty\prec x\prec\infty$.

f. $y=\sin x^2$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$P_n(x)=x^2-\frac{(x^3)^2}{3!}+\frac{(x^5)^2}{5!}+\ldots+\frac{(-1)^n(x^{2n+1})^2}{(2n+1)!}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2(2n+1)}}{(2n+1)!}=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}+\ldots+\frac{(-1)^n x^{2(2n+1)}}{(2n+1)!}$$

Interval of convergence:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(2(n+1)+1)}}{(2(n+1)+1)!}\frac{(2n+1)!}{(-1)^n x^{2(2n+1)}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{4n+6}}{(2n+3)(2n+2)(2n+1)2n!}\frac{(2n+1)2n!}{(-1)^n x^{4n+2}}\right|$$

$$\lim_{n\to\infty}\left|\frac{-x^4}{(2n+3)(2n+2)}\right|=0\prec 1$$

The series is convergent for $-\infty \prec x \prec \infty$.

g. $y = \frac{e^{x^3} - 1}{x^2}$

$$\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}\right)^3 - 1}{x^2}$$

$$\frac{x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \ldots + \frac{x^{3n}}{n!}}{x^2}$$

$$x + \frac{x^4}{2!} + \frac{x^7}{3!} + \ldots + \frac{x^{3n-2}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^{3n-2}}{n!}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty} \left| \frac{x^{3(n+1)-2}}{(n+1)!} \frac{n!}{x^{3n-2}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{3n+1}}{(n+1)n!} \frac{n!}{x^{3n-2}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{(n+1)} \frac{1}{x^{-2}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^3}{(n+1)} \right|$$

The series is convergent for $-\infty \prec x \prec \infty$.

h. $y = \cos x^2$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n})}{(2n)!}$$

$$P_n(x) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^4)^2}{4!} - \frac{(x^6)^2}{6!} + \frac{(x^8)^2}{8!} - \frac{(x^{10})^2}{10!} + \ldots + \frac{(-1)^n (x^{2n})^2}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots + \frac{(-1)^n x^{4n}}{(2n)!}$$

Interval of convergence (ratio test):

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{4(n+1)}}{(2(n+1))!}\frac{(2n)!}{(-1)^n x^{4n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)x^{4n+4}}{(2n+2)(2n+1)(2n)!}\frac{(2n)!}{x^{4n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{-x^4}{(2n+2)(2n+1)}\right| = 0 \prec 1$$

The series is convergent for $-\infty \prec x \prec \infty$.

i. $y=\ln(1+x)$

$$P_n(x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

Interval of convergence:

$$\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{n+1}\frac{n}{(-1)^{n-1}x^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{-xn}{n+1}\right| = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty}\left|\frac{-x}{1}\right| = |-x|$$

$$-1 \prec x \prec 1$$

If $x=-1$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=0}^{\infty}(-1)^{2n-1}\frac{1}{n} = \sum_{n=0}^{\infty}\frac{-1}{n}$$

P-series with $p \le 1$ (Divergent)

If $x=1$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(1)^n}{n} = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{-1}{n} = 0$$

Alternating series (Convergent)

The series $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}x^n}{n}$ is convergent for $-1 \prec x \le 1$.